

(m, n) –String in (p, q) –String and (p, q) –Five Brane Background

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ABSTRACT: We study dynamics of (m, n) –string in (p, q) –five brane and (p, q) –string background. We determine world-volume stress energy tensor and we analyze the dependence of the string's dynamics on the values of the charges (m, n) and the value of the angular momentum.

KEYWORDS: D-brane, Supergravity Background.

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1. Introduction and Summary

Low energy effective actions of superstring theories have reach spectrum of solutions that preserve some fractions of supersymmetry, for review see for example [1, 2, 3, 4]. These objects have property that they are sources of various form fields that are presented in supergravity theories. Further, fundamental string, D-brane and NS5-brane solutions preserve one half of the space-time supersymmetries and can be considered as the building block of other solutions. For example, taking intersection of these configurations we get backgrounds that preserve some fractions of supersymmetry [5]. Another possibility is to generate new solutions using U-duality symmetry of M-theory (For review see for example [6]) which is basically the symmetry of M-theory on its maximally supersymmetric toroidal compactifications. For example, M-theory compactified on two torus possesses $SL(2, Z)$ symmetry which turns out to be non-perturbative $SL(2, Z)$ duality of type IIB theory. More precisely, it is well known that low effective action of type IIB supergravity written in Einstein frame is invariant under $SL(2, R)$ duality. Special case of $SL(2, R)$ transformation is S-duality transformation that roughly speaking transforms theory at weak coupling to the strong coupling. The fact that the type IIB supergravity action is invariant under this symmetry suggests the possibility how to generate new supergravity solutions when we apply $SL(2, R)$ rotation on known supergravity solutions, as for example fundamental string or NS5-brane backgrounds. Such a procedure was firstly used in a famous paper [7] where the manifestly $SL(2, R)$ covariant supergravity solution corresponding to (p, q) -string was found. The extension of this analysis to the case of NS5-brane was performed in [8] when $SL(2, Z)$ covariant expression for supergravity solutions corresponding to (p, q) -five

brane was derived ¹. These backgrounds are very interesting and certainly deserve to be studied further. In particular, it is well known that the continuous classical symmetry group $SL(2, R)$ of type IIB supergravity cannot be a symmetry of the full string theory when non-perturbative effects break it to a discrete subgroup $SL(2, Z)$. To see this more clearly note that fundamental string carries one unit of NSNS two form charge and hence this charge has to be quantized in integer units. On the other hand $SL(2, R)$ transformations maps a fundamental string into a string with d -units of this charge where d is an entry of $SL(2, R)$ matrix. From this result we conclude that d -has to be integer. In the similar way we can argue that $SL(2, R)$ symmetry of the low energy effective action has to be broken to its $SL(2, Z)$ subgroup when fundamental string is mapped under this duality to (p, q) -string that carries p charge of NSNS-two form and q charge of Ramond Ramond two form [9]. It was also shown in [9] that the type IIB string effective action together with (p, q) -string action is covariant under $SL(2, R)$ transformations. However the fact that (p, q) string has to map to another (p', q') -string where p', q' are integers suggests that the full symmetry group of the combined action breaks to $SL(2, Z)$. On the other hand solutions found in [7, 8] were determined using the $SL(2, R)$ matrices so that it is interesting to analyze the problem of (m, n) -string probe in such a background and this is precisely the aim of this paper.

We begin with the D1-brane action that we rewrite into a manifestly covariant $SL(2, Z)$ form, for related analysis see [10] and for very elegant formulation of manifestly $SL(2, Z)$ covariant superstring, see [11, 12]. Now using the fact that (p, q) -five and fundamental string solutions were derived using $SL(2, R)$ transformations we can map the problem of the dynamics of (m, n) -string in this background to the problem of the analysis of (m', n') -string in the original NS5-brane and fundamental string background with crucial exception that the harmonic functions that define these solutions have constant factors that differ from the factors that define NS5-brane and fundamental string solutions. It is also important to stress that now (m', n') are not integers but depend on p, q and also on asymptotic values of dilaton and Ramond-Ramond zero form. We mean that this is not quite satisfactory resort and one can ask the questions whether it would be possible to find (p, q) -string and five brane backgrounds that are derived from the NS5-brane and fundamental string background through manifest $SL(2, Z)$ transformations when in probe (m, n) -string will transform in an appropriate way. This problem is currently under study and we return to it in near future. We rather focus on the dynamics of the probe (m, n) -string in the backgrounds [7, 8], following very nice analysis introduced in [14]. Using manifest $SL(2, Z)$ covariant formulation of a probe (m, n) -string we can analyze the time evolution of homogeneous time dependent string in given background. We determine components of the world-sheet stress energy tensor and study its time evolution. The properties of this stress energy tensor and the dynamics of the probe depends on the values of m, n and hence our results can be considered as the generalization of the analysis performed in [14].

As the next step we analyze the dynamics of the probe (m, n) -string in the background of (p, q) -macroscopic string. Thanks to the form of the solution [7] we formulate this

¹For non-extremal form of these solutions, see [17].

problem as the analysis of the dynamics of (m', n') -string in the background of fundamental string. This problem was studied previously in [15] but we focus on different aspect of the dynamics of the probe. Explicitly we will be interested in the behavior of the probe where the difference between its energy and the rest energy is small. We find that the potential is flat which is in agreement with the fact that the string probe in the fundamental string background can form marginal bound state with the strings that are sources of this background. We also analyze the situation with non-zero angular momentum and we find that there is a potential barrier that does not allow the probe string to move towards to the horizon. These results are in agreement with the analysis performed in [15].

The organization of this paper is as follows. In the next section (2) we review $SL(2, R)$ duality of type IIB low energy effective action. We also introduce manifestly $SL(2, R)$ covariant action for (m, n) -string. In section (3) we study the dynamics of this string in the background of (p, q) -five brane. Finally in section (4) we study dynamics of (m, n) -string in the background of (p, q) -string.

2. $SL(2, R)$ -Covariance of type IIB Low Energy Effective Action

The type IIB theory has two three-form field strengths $H = dB, F = dC^{(2)}$, where H corresponds to NSNS three form while F belongs to RR sector and does not couple to the usual string world-sheet. Type IIB theory has also two scalar fields that can be combined into a complex field $\tau = \chi + ie^{-\Phi}$. The dilaton Φ is in the NSNS sector while χ belongs to the RR sector. The other bose fields are the metric $g_{\mu\nu}$ and self-dual five form field strength F_5 that we set zero in this paper. Then it is possible to write down a covariant form of the bosonic part of type IIB effective action

$$S_{IIB} = \frac{1}{2\tilde{\kappa}_{10}^2} \int d^{10}x \sqrt{-g} (R + \frac{1}{4} \text{Tr}(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}^{-1}) - \frac{1}{12} \mathbf{H}_{\mu\nu\sigma}^T \mathcal{M} \mathbf{H}^{\mu\nu\sigma}) , \quad (2.1)$$

where $\tilde{\kappa}_{10}^2 = \frac{1}{4\pi}(4\pi^2\alpha')^4$ and where we have combined $B, C^{(2)}$ into

$$\mathbf{H} = d\mathbf{B} = \begin{pmatrix} dB \\ dC^{(2)} \end{pmatrix} , \quad (2.2)$$

and where

$$\mathcal{M} = e^\Phi \begin{pmatrix} \tau\tau^* & \chi \\ \chi & 1 \end{pmatrix} = e^\Phi \begin{pmatrix} \chi^2 + e^{-2\Phi} & \chi \\ \chi & 1 \end{pmatrix} . \quad (2.3)$$

The action (2.1) has manifest invariance under the global $SL(2, R)$ transformation

$$\hat{\mathcal{M}} = \Lambda \mathcal{M} \Lambda^T , \hat{\mathbf{B}} = (\Lambda^T)^{-1} \mathbf{B} , \quad (2.4)$$

where

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \quad (2.5)$$

It is well known that all string theories contain fundamental string and magnetic dual NS5-brane as solutions of the equations of motion of its low energy effective actions. Using a manifest $SL(2, R)$ covariance of type IIB effective action it is possible to derive solutions corresponding to (p, q) -five brane [8] and fundamental string [7]. It will be certainly interesting to analyze properties of given background with the help of the appropriate probe which will be probe (m, n) -string. For that reason we introduce a manifestly covariant form of (m, n) -string action.

2.1 (m, n) -String Action

In this section we formulate the action for the (m, n) -string. Even if such a formulation is well known [9, 10, 11, 12, 13] we derive this action in a slightly different way with the help of the Hamiltonian formalism which will be also useful for the analysis of the dynamics of probe (m, n) -string in (p, q) -five and (p, q) -string background.

To begin with we introduce an action for n coincident D1-branes in general background

$$\begin{aligned}
S = & -nT_{D1} \int d\tau d\sigma e^{-\Phi} \sqrt{-\det \mathbf{A}} + \\
& + nT_{D1} \int d\tau d\sigma ((b_{\tau\sigma} + 2\pi\alpha' \mathcal{F}_{\tau\sigma})\chi + c_{\tau\sigma}) , \\
\mathbf{A}_{\alpha\beta} = & G_{MN} \partial_\alpha x^M \partial_\beta x^N + 2\pi\alpha' \mathcal{F}_{\alpha\beta} + B_{MN} \partial_\alpha x^M \partial_\beta x^N , \\
\mathcal{F}_{\alpha\beta} = & \partial_\alpha A_\beta - \partial_\beta A_\alpha ,
\end{aligned} \tag{2.6}$$

where $x^M, M, N = 0, 1, \dots, 9$ are embedding coordinates of D1-brane in the background that is specified by the metric G_{MN} and NSNS two form $B_{MN} = -B_{NM}$ together with Ramond-Ramond two form $C_{MN}^{(2)} = -C_{NM}^{(2)}$. Note that we use capital letter G_{MN} for the string frame metric while g_{MN} corresponds to the Einstein frame metric. We further consider background with non-trivial dilaton Φ and RR zero form χ . Further, $\sigma^\alpha = (\tau, \sigma)$ are world-sheet coordinates and $b_{\tau\sigma}, c_{\tau\sigma}$ are pull-backs of B_{MN} and C_{MN} to the world-volume of D1-brane. Explicitly,

$$b_{\alpha\beta} \equiv B_{MN} \partial_\alpha x^M \partial_\beta x^N , \quad c_{\tau\sigma} = C_{MN}^{(2)} \partial_\tau x^M \partial_\sigma x^N . \tag{2.7}$$

Finally $T_{D1} = \frac{1}{2\pi\alpha'}$ is D1-brane tension and $A_\alpha, \alpha = \tau, \sigma$ is two dimensional gauge field that propagates on the world-sheet of D1-brane.

It is useful to rewrite the action (2.6) into the form

$$\begin{aligned}
S = & -nT_{D1} \int d\tau d\sigma e^{-\Phi} \sqrt{-\det g - (2\pi\alpha' \mathcal{F}_{\tau\sigma} + b_{\tau\sigma})^2} + \\
& + nT_{D1} \int d\tau d\sigma ((b_{\tau\sigma} + 2\pi\alpha' \mathcal{F}_{\tau\sigma})\chi + c_{\tau\sigma}) ,
\end{aligned} \tag{2.8}$$

where $g_{\alpha\beta} = G_{MN} \partial_\alpha x^M \partial_\beta x^N, \det g = g_{\tau\tau} g_{\sigma\sigma} - (g_{\tau\sigma})^2$. Now we proceed to the Hamiltonian formulation of the theory defined by the action (2.8). First of all we derive conjugate

momenta to x^M and A_α from (2.8)

$$\begin{aligned}
p_M &= \frac{\delta L}{\delta \partial_\tau x^M} = nT_{D1} \frac{e^{-\Phi}}{\sqrt{-\det g - (2\pi\alpha' F_{\tau\sigma} + b_{\tau\sigma})^2}} (G_{MN} \partial_\alpha x^N g^{\alpha\tau} \det g + \\
&\quad + (2\pi\alpha' F_{\tau\sigma} + b_{\tau\sigma}) B_{MN} \partial_\sigma x^N) + nT_{D1} (\chi B_{MN} \partial_\sigma x^N + C_{MN}^{(2)} \partial_\sigma x^N) , \\
\pi^\sigma &= \frac{\delta L}{\delta \partial_\tau A_\sigma} = \frac{ne^{-\Phi} (2\pi\alpha' F_{\tau\sigma} + b_{\tau\sigma})}{\sqrt{-\det g - (2\pi\alpha' F_{\tau\sigma} + b_{\tau\sigma})^2}} + n\chi , \quad \pi^\tau = \frac{\delta L}{\delta \partial_\tau A_\tau} \approx 0
\end{aligned} \tag{2.9}$$

and hence

$$\begin{aligned}
\Pi_M &\equiv p_M - \frac{\pi^\sigma}{(2\pi\alpha')} B_{MN} \partial_\sigma x^N - nT_{D1} C_{MN}^{(2)} \partial_\sigma x^N = \\
&= nT_{D1} \frac{e^{-\Phi}}{\sqrt{-\det g - (2\pi\alpha' F_{\tau\sigma} + b_{\tau\sigma})^2}} G_{MN} \partial_\alpha x^N g^{\alpha\tau} \det g .
\end{aligned} \tag{2.10}$$

Using these relations it is easy to see that the bare Hamiltonian is equal to

$$H_B = \int d\sigma (p_M \partial_\tau x^M + \pi^\sigma \partial_\tau A_\sigma - \mathcal{L}) = \int d\sigma \pi^\sigma \partial_\sigma A_\tau \tag{2.11}$$

while we have three primary constraints

$$\begin{aligned}
\pi^\tau &\approx 0 , \quad \mathcal{H}_\sigma \equiv p_M \partial_\sigma x^M \approx 0 , \\
\mathcal{H}_\tau &\equiv \frac{1}{T_{D1}} \Pi_M G^{MN} \Pi_N + T_{D1} \left(n^2 e^{-2\Phi} + (\pi^\sigma - n\chi)^2 \right) g_{\sigma\sigma} \approx 0 .
\end{aligned} \tag{2.12}$$

Including these primary constraints to the definition of the Hamiltonian we obtain an extended Hamiltonian in the form

$$H = \int d\sigma (\lambda_\tau \mathcal{H}_\tau + \lambda_\sigma \mathcal{H}_\sigma - A_\tau \partial_\sigma \pi^\sigma + v_\tau \pi^\tau) , \tag{2.13}$$

where $\lambda_\tau, \lambda_\sigma, v_\tau$ are Lagrange multipliers corresponding to the primary constraints $\mathcal{H}_\tau \approx 0, \mathcal{H}_\sigma \approx 0, \pi^\tau \approx 0$. Now we have to check the stability of all constraints. The requirement of the preservation of the primary constraint $\pi^\tau \approx 0$ implies the secondary constraint

$$\mathcal{G} = \partial_\sigma \pi^\sigma \approx 0 . \tag{2.14}$$

In case of the constraints $\mathcal{H}_\tau, \mathcal{H}_\sigma$ we can easily show in the same way as in [16] that the constraints $\mathcal{H}_\tau, \mathcal{H}_\sigma$ are first class constraints and hence they are preserved during the time evolution.

An action for (m, n) -string is derived when we fix the gauge generated by \mathcal{G} with the gauge fixing function $A_\sigma = \text{const}$. Then the fixing of the gauge implies that $\pi^\sigma = f(\tau)$ but the equation of motion for π^σ implies that $\partial_\tau \pi^\sigma = 0$ and hence $\pi^\sigma = m$, where m is integer

that counts the number of fundamental string bound to n D1-branes. After this partial gauge fixing the Hamiltonian density has the form

$$\mathcal{H}_{(m,n)} = \int d\sigma (\lambda_\tau \mathcal{H}_\tau + \lambda_\sigma \mathcal{H}_\sigma) . \quad (2.15)$$

In order to find (m,n) -string action we derive Lagrangian density corresponding to the Hamiltonian (2.15). Explicitly, from (2.15) we obtain equations of motion for x^M

$$\partial_\tau x^M = \{x^M, H_{(m,n)}\} = 2\lambda_\tau \frac{1}{T_{D1}} G^{MN} \Pi_N + \lambda_\sigma \partial_\sigma x^M \quad (2.16)$$

and hence

$$\begin{aligned} \mathcal{L}_{(m,n)} &= p_M \partial_\tau x^M - \mathcal{H}_{(m,n)} = \\ &= \frac{1}{2\pi\alpha'} \left(\frac{1}{4\lambda_\tau} (g_{\tau\tau} - 2\lambda_\sigma g_{\tau\sigma} + \lambda_\sigma^2 g_{\sigma\sigma}) - \lambda_\tau (n^2 e^{-2\Phi} + (m - n\chi)^2) g_{\sigma\sigma} + m b_{\tau\sigma} + n c_{\tau\sigma} \right) . \end{aligned} \quad (2.17)$$

As the final step we solve the equation of motion for λ_τ and λ_σ that follow from (2.17) and we obtain

$$\lambda_\sigma = \frac{g_{\tau\sigma}}{g_{\sigma\sigma}} , \quad \lambda_\tau = \frac{1}{2g_{\sigma\sigma} \sqrt{n^2 e^{-2\Phi} + (m - n\chi)^2}} \sqrt{-\det g} . \quad (2.18)$$

Inserting this result into the Lagrangian density (2.17) we obtain the action in manifestly covariant $SL(2, R)$ form

$$\begin{aligned} S &= -T_{D1} \int d\tau d\sigma (\sqrt{\mathbf{m}^T \mathcal{M}^{-1} \mathbf{m}} \sqrt{-\det g_{MN} \partial_\alpha x^M \partial_\beta x^N} \\ &\quad + T_{D1} \int d\tau d\sigma \mathbf{m}^T \mathbf{B}_{MN} \partial_\tau x^M \partial_\sigma x^N , \end{aligned} \quad (2.19)$$

where

$$\mathbf{m} = \begin{pmatrix} m \\ n \end{pmatrix} , \quad \mathbf{B}_{MN} = \begin{pmatrix} B_{MN} \\ C_{MN}^{(2)} \end{pmatrix} , \quad (2.20)$$

and where $g_{MN} = e^{-\Phi/2} G_{MN}$ is Einstein-frame metric. Since $\hat{\mathbf{B}} = (\Lambda^T)^{-1} \mathbf{B}$ we obtain that \mathbf{m} transforms as

$$\hat{\mathbf{m}} = \Lambda \mathbf{m} \quad (2.21)$$

in order the action (2.19) to be manifestly $SL(2, R)$ covariant. On the other hand since m, n count the number of fundamental strings and D1-branes and hence have to be integers we find that the non-perturbative duality group of type IIB superstring theory is $SL(2, Z)$ which will have an important consequence for the analysis of the dynamics of (m, n) -string in (p, q) -five brane and (p, q) -fundamental string background.

3. (m, n) -String in the Background of (p, q) -Five Brane

We would like to analyze the dynamics of (m, n) -string in the background of (p, q) -five brane that has the form [8]

$$\begin{aligned} ds_E^2 &= (1 + \frac{Q_{(p,q)}}{r^2})^{-1/4} \eta_{\mu\nu} dx^\mu dx^\nu + (1 + \frac{Q_{(p,q)}}{r^2})^{3/4} dx^m dx^m , \\ \lambda &= \chi + ie^{-\Phi} = \frac{\chi_0 \Delta_{(p,q)} A_{(p,q)} + pqe^{-\Phi_0} (A_{(p,q)} - 1) + i\Delta_{(p,q)} A_{(p,q)}^{1/2} e^{-\Phi_0}}{p^2 e^{-\Phi_0} + A_{(p,q)} e^{\Phi_0} (\chi_0 p + q)^2} , \\ H &= dB = 2p(2\pi\alpha')^2 \epsilon_3 , \quad F = dC_2 = 2q(2\pi\alpha')^2 \epsilon_3 , \end{aligned} \quad (3.1)$$

where

$$Q_{(p,q)} = \sqrt{\Delta_{(p,q)} 2\pi\alpha'} = \sqrt{e^{-\Phi_0} p^2 + (q + p\chi_0)^2 e^{\Phi_0}} 2\pi\alpha' , \quad (3.2)$$

and where ϵ_3 is volume form of the three sphere when we express the line element of the transverse space $dx_m dx^m$ as $dx_m dx^m = dr^2 + r^2 d\Omega_3$. Note also that $x^\mu, \mu = 0, \dots, 5$ label directions along the world-volume of (p, q) -five brane. Further $A_{(p,q)}$ is defined as

$$A_{(p,q)} = \left(1 + \frac{Q_{(p,q)}}{r^2}\right)^{-1} , \quad (3.3)$$

and ds_E^2 means that this line element is expressed in Einstein frame metric. Let us now consider probe (m, n) -string action (2.19) in given background. The analysis of this problem simplifies considerably when we realize how the solution (3.1) was determined. Following [8] and [7] we introduce $SL(2, R)$ matrix

$$\Lambda = \Delta_{(p,q)}^{-1/2} \begin{pmatrix} e^{-\Phi_0} p + \chi_0 e^{\Phi_0} (q + p\chi_0) & -(q + p\chi_0) + \chi_0 p \\ e^{\Phi_0} (q + p\chi_0) & p \end{pmatrix} , \quad (3.4)$$

where

$$\Delta_{(p,q)} = e^{-\Phi_0} p^2 + (q + p\chi_0)^2 e^{\Phi_0} , \quad (3.5)$$

and where χ_0 and Φ_0 are asymptotic values of fields Φ and χ . Note that the inverse matrix has the form

$$\Lambda^{-1} = \Delta_{(p,q)}^{-1/2} \begin{pmatrix} p & q \\ -e^{\Phi_0} (q + p\chi_0) & e^{-\Phi_0} p + \chi_0 e^{\Phi_0} (q + p\chi_0) \end{pmatrix} .$$

Now with the help of this matrix we can write \mathcal{M} as [8]

$$\mathcal{M} = \Lambda(p, q) \begin{pmatrix} \sqrt{A_{(p,q)}} & 0 \\ 0 & \frac{1}{\sqrt{A_{(p,q)}}} \end{pmatrix} \Lambda^T(p, q) \quad (3.6)$$

so that

$$\mathbf{m}^T \mathcal{M}^{-1} \mathbf{m} = m'^2 \frac{1}{\sqrt{A_{(p,q)}}} + n'^2 \sqrt{A_{(p,q)}} ,$$

where

$$\mathbf{m}' = \begin{pmatrix} m' \\ n' \end{pmatrix} = \Lambda^{-1}(p, q) \mathbf{m} = \Delta_{(p,q)}^{-1/2} \begin{pmatrix} pm + qn \\ e^{\Phi_0}(q + p\chi_0)(-m + n\chi_0) + e^{-\Phi_0}pn \end{pmatrix}. \quad (3.7)$$

It is interesting that for the special values of m, n equal to

$$m = -q, n = p \quad (3.8)$$

we obtain

$$\mathbf{m}' = \Delta^{-1/2} \begin{pmatrix} 0 \\ e^{\Phi_0}(q + p\chi_0)^2 + e^{-\Phi_0}p^2 \end{pmatrix} = \begin{pmatrix} 0 \\ \Delta_{(p,q)}^{1/2} \end{pmatrix}. \quad (3.9)$$

Since $m' = 0$ we can interpret this configuration as a pure D1-brane which however does not have integer charge. We also see from (3.7) that in order to find configuration with $n' = 0$ we have to demand that $\Phi_0 = 0 = \chi_0$ and set $m = p, n = q$

$$\mathbf{m}' = \Delta^{-1/2} \begin{pmatrix} p^2 + q^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{p^2 + q^2} \\ 0 \end{pmatrix}. \quad (3.10)$$

Generally we see that the action for the probe (m, n) -string in (p, q) -five brane background is equivalent to the action of (m', n') -string in NS5-brane background with an important exception that the harmonic function has the factor $Q_{(p,q)}$ (3.2) instead of the standard one that corresponds to the number of NS5-branes. Note also that m', n' depend on m, n, p, q and moduli Φ_0 and χ_0 as follows from (3.7).

Let us now return to the analysis of dynamics of probe (m, n) -string in this background. It is convenient to impose the static gauge

$$x^0 = \tau, \quad x^1 = \sigma \quad (3.11)$$

and introduce spherical coordinates in the transverse space \mathbf{R}^4

$$x^1 = r \cos \psi, x_1 = r \sin \psi \cos \theta, x^3 = r \sin \psi \sin \theta \cos \phi, x^4 = r \sin \psi \sin \theta \sin \phi \quad (3.12)$$

so that volume element of Ω_3 is equal to

$$d\Omega_3 = \sin^2 \psi \sin \theta d\psi \wedge d\theta \wedge d\phi. \quad (3.13)$$

Using these equations we obtain that we have following components of RR and NSNS two forms

$$B_{\psi\phi} = 2p(2\pi\alpha')^2 \sin^2 \phi \cos \theta, \quad C_{\psi\phi}^{(2)} = 2q(2\pi\alpha')^2 \sin^2 \psi \cos \theta. \quad (3.14)$$

Now we would like to derive the components of the stress energy tensor $T_{\alpha\beta}$ for the gauge fixed theory. To do this we temporary replace fixed two dimensional metric $\eta_{\alpha\beta}$ with two dimensional metric $\gamma_{\alpha\beta}$ and write the gauge fixed action in the form

$$S_{fixed} = -T_{D1} \int d\tau d\sigma (\sqrt{\mathbf{m}^T \mathcal{M}^{-1} \mathbf{m}} A_{(p,q)}^{1/4} \sqrt{-\det \mathbf{A}_{\alpha\beta}} + S_{WZ}), \quad (3.15)$$

where

$$\mathbf{A}_{\alpha\beta} = \gamma_{\alpha\beta} + \frac{1}{A_{(p,q)}} \delta_{mn} \partial_\alpha x^m \partial_\beta x^n + \delta_{\alpha\beta} \partial_\alpha x^\alpha \partial_\beta x^\beta , \quad (3.16)$$

where $x^\alpha, \alpha = 1, \dots, 5$ label coordinates along the world-volume of (p, q) -five brane. Then we define components of two dimensional stress energy tensor as

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-\det\gamma}} \frac{\delta S_{fixed}}{\delta \gamma^{\alpha\beta}} = -\frac{T_{D1}}{\sqrt{-\det\gamma}} \gamma_{\alpha\gamma} (\mathbf{A}^{-1})^{\gamma\delta} \gamma_{\delta\beta} \sqrt{-\det\mathbf{A}} \sqrt{\mathbf{m}^T \mathcal{M}^{-1} \mathbf{m}} A_{(p,q)}^{1/4} . \quad (3.17)$$

Now we return back to the flat metric $\gamma_{\alpha\beta} \rightarrow \eta_{\alpha\beta}$ and consider pure time-dependent ansatz. As a result we obtain following components of the world-sheet stress energy tensor

$$\begin{aligned} T_{\tau\tau} &= \frac{T_{D1} \sqrt{\mathbf{m}^T \mathcal{M}^{-1} \mathbf{m}} A_{(p,q)}^{1/4}}{\sqrt{1 - \frac{1}{A_{(p,q)}} \partial_\tau x^m \partial_\tau x_m - \partial_\tau x^\alpha \partial_\tau x_\alpha}} , \quad T_{\tau\sigma} = 0 , \\ T_{\sigma\sigma} &= -T_{D1} \sqrt{\mathbf{m}^T \mathcal{M}^{-1} \mathbf{m}} A_{(p,q)}^{1/4} \sqrt{1 - \frac{1}{A_{(p,q)}} \partial_\tau x^m \partial_\tau x_m - \partial_\tau x^\alpha \partial_\tau x_\alpha} \end{aligned} \quad (3.18)$$

that are generalization of the components of the stress energy tensor of Dp-brane moving in NS5-brane background as were found in [14].

3.1 Gauge Fixing in Hamiltonian formalism

Now we proceed to the analysis of dynamics of probe (m, n) -string in (p, q) -five brane background. It turns out that it is useful to perform this analysis in the canonical approach when we impose the static gauge using two gauge fixing functions

$$\mathcal{G}_\tau = x^0 - \tau \approx 0 , \quad \mathcal{G}_\sigma = x^1 - \sigma \approx 0 . \quad (3.19)$$

These constraints have non-zero Poisson brackets with $\mathcal{H}_\tau \approx 0$, $\mathcal{H}_\sigma \approx 0$ so that they are the second class constraints. As a result $\mathcal{H}_\tau, \mathcal{H}_\sigma$ vanish strongly and can be solved for p_0 and p_1 respectively, where we can relate $-p_0$ with the Hamiltonian density of gauge fixed theory \mathcal{H}_{fix} . To see this note that the action has the form

$$S = \int d\tau d\sigma (p_M \partial_\tau x^M - \mathcal{H}) = \int d\tau d\sigma (p_i \partial_\tau x^i + p_0) = \int d\tau d\sigma (p_i \partial_\tau x^i - \mathcal{H}_{fix}) . \quad (3.20)$$

Now from $\mathcal{H}_\sigma = 0$ we obtain $p_1 = -(p_i \partial_\sigma x^i)$ and from \mathcal{H}_τ we find

$$\begin{aligned} \mathcal{H}_{fix} &= \sqrt{-g_{00} (\Pi_1 g^{11} \Pi_1 + \Pi_i g^{ij} \Pi_j + T_{D1}^2 (m'^2 e^\Phi + n'^2 e^{-\Phi}) (g_{11} + g_{ij} \partial_\sigma x^i \partial_\sigma x^j))} \\ &\quad - \frac{1}{2\pi\alpha'} \mathbf{m}^T \mathbf{B}_{0M} \partial_\sigma x^M , \end{aligned} \quad (3.21)$$

where $i, j = 2, \dots, 9$. The analysis simplifies further when we presume that the embedding modes depend on τ only so that the Hamiltonian density (3.21) reduces into

$$\begin{aligned} \mathcal{H}_{fix}^2 = & A_{(p,q)}^{1/4} \left(p^\alpha p_\alpha A_{(p,q)}^{-1/4} + A_{(p,q)}^{3/4} (p_r^2 + \frac{1}{r^2} p_\psi^2 + \frac{1}{r^2 \sin^2 \psi} p_\theta^2 + \frac{1}{r^2 \sin^2 \psi \sin^2 \theta} p_\phi^2) + \right. \\ & \left. + T_{D1}^2 (m'^2 + n'^2 A_{(p,q)}) A_{(p,q)}^{-1/4} \right) \equiv \mathcal{K} , \end{aligned} \quad (3.22)$$

where $p_\alpha, \alpha = 2, 3, 4, 5$ denote momenta along the world-volume of (p, q) -five branes. Since they are conserved we restrict ourselves to the case when $p_\alpha = 0$. At the same time we find that p_ϕ is conserved as well and we denote this constant as $p_\phi = L$. On the other hand the equations of motion for θ, p_θ have the form

$$\begin{aligned} \dot{\theta} = \{ \theta, H_{fix} \} &= \frac{A_{(p,q)} p_\theta}{r^2 \sin^2 \psi \sqrt{\mathcal{K}}} , \\ \dot{p}_\theta = \{ p_\theta, H_{fix} \} &= \frac{A_{(p,q)} \sin \theta \cos \theta}{r^2 \sin^2 \psi \sin^3 \theta \sqrt{\mathcal{K}}} p_\psi^2 . \end{aligned} \quad (3.23)$$

We see that this equation has the solution when $\theta = \frac{\pi}{2}$ and $p_\theta = 0$. In the same way we find that $p_\psi = 0, \psi = \frac{\pi}{2}$ solve the equations of motion. Finally we proceed to the analysis of the time evolution of r . The equation of motion for r gives

$$\dot{r} = \{ r, H_{fix} \} = \frac{A_{(p,q)} p_r}{\sqrt{\mathcal{K}}} . \quad (3.24)$$

To proceed further we use the fact that the Hamiltonian density \mathcal{H}_{fix} is conserved and we denote its constant value as E . Then we can solve $\mathcal{H}_{fix} = E$ for p_r as

$$p_r = \sqrt{\frac{E^2 - \frac{A_{(p,q)} L^2}{r^2} - T_{D1}^2 (m'^2 + n'^2 A_{(p,q)})}{A_{(p,q)}}} \quad (3.25)$$

so that from (3.24) we obtain

$$\dot{r}^2 = A_{(p,q)} - \frac{A_{(p,q)}^2}{E^2} \left(\frac{L^2}{r^2} + T_{D1}^2 n'^2 \right) - \frac{A_{(p,q)} T_{D1}^2}{E^2} m'^2 . \quad (3.26)$$

As the check note that the first two terms on the right side in (3.26) coincide with the expression that governs the dynamic of Dp-brane in NS5-brane background [14] while the last one that is proportional to m'^2 corresponds to the dynamics of the fundamental string in this background. For the next purposes we also determine the equation of motion for ϕ

$$\dot{\phi} = \{ \phi, H_{fix} \} = \frac{A_{(p,q)} L}{r^2 E} . \quad (3.27)$$

3.2 The Case $L = 0$

We firstly consider the case of the vanishing angular momentum $p_\theta = L = 0$. Then the equation (3.27) implies that ϕ is a constant while the (3.26) has the form

$$\dot{r}^2 = A_{(p,q)} - \frac{A_{(p,q)}^2 T_{D1}^2}{E^2} n'^2 - \frac{A_{(p,q)} T_{D1}^2}{E^2} m'^2 . \quad (3.28)$$

Now we will analyze this expression in more details. First of all the solution of this equation is restricted to the region where the right side is non-negative. Since

$$A_{(p,q)} = \left(1 + \frac{Q_{(p,q)}}{r^2} \right)^{-1} . \quad (3.29)$$

we obtain

$$\frac{Q_{(p,q)}}{r^2} > \frac{T_{D1}^2 n'^2}{E^2 (1 - \frac{T_{D1}^2}{E^2} m'^2)} - 1 . \quad (3.30)$$

Note that for $m' = 0$ this result agrees with the result derived in [14]. We see that this condition is empty when

$$E^2 > T_{D1}^2 (n'^2 + m'^2) \quad (3.31)$$

that has clear physical meaning. It corresponds to the situation when the total energy is greater then the asymptotic tension of (m', n') string and given string can escape to infinity. Note that for $E^2 < T_{D1}^2 (n'^2 + m'^2)$ the (m', n') string cannot escape the attraction from five-brane.

We also determine the components of the stress energy tensor (3.18) for this configuration. Using (3.28) we easily find

$$\begin{aligned} T_{\tau\tau} &= \frac{T_{D1} \sqrt{m'^2 + n'^2 A_{(p,q)}}}{\sqrt{1 - \frac{1}{A_{(p,q)}} \dot{r}^2}} = E , \quad T_{\tau\sigma} = 0 , \\ T_{\sigma\sigma} &= -T_{D1} \sqrt{m'^2 + n'^2 A_{(p,q)}} \sqrt{1 - \frac{1}{A_{(p,q)}} \dot{r}^2} = -\frac{T_{D1}^2}{E} (m'^2 + n'^2 A_{(p,q)}) . \end{aligned} \quad (3.32)$$

From $T_{\sigma\sigma} = \mathcal{P}$ we see that the contribution from D1-brane to the pressure goes to zero when we approach the core of the five-brane background while the string like contribution is constant. This is an analogue of the well known fact that the fundamental string can make the bound state with NS5-brane.

Let us now consider such an energy interval when the entire trajectory in the region when $Q_{(p,q)} \gg r^2$. Then the equation for \dot{r} has the form

$$\dot{r}^2 = \frac{r^2}{Q_{(p,q)}} \left(1 - \frac{T_{D1}^2 m'^2}{E^2} \right) - \frac{r^4}{Q_{(p,q)}^2} \frac{T_{D1}^2}{E^2} n'^2 \quad (3.33)$$

that has solution

$$r = \frac{1}{n'} \sqrt{Q_{(p,q)} \frac{E^2}{T_{D1}^2} - m'^2} \frac{1}{\cosh \sqrt{\left(1 - \frac{T_{D1}^2}{E^2} m'^2\right) \frac{1}{Q_{(p,q)}} t}} , \quad (3.34)$$

where we chosen the initial condition that for $t = 0$ (m', n')–string is at the point of the maximal value corresponding to $\dot{r} = 0$. From the previous expression we see that this result is valid in case of $m' = 0$. On the other hand the case $n' = 0$ has to be analyzed separately in the equation (3.33) and we obtain the result

$$r = r_0 e^{\pm \sqrt{\frac{1}{Q_{(p,q)}} \left(1 - \frac{T_{D1}^2}{E^2} m'^2\right) t}} , \quad (3.35)$$

where the $-$ sign corresponds to m' –string moving towards to the world-volume of five brane while $+$ corresponds to the situation when m' –string leaves it. Again, this result is the manifestation of the fact that the fundamental string can form marginal bound state with NS5-brane. However in our case this situation is not so clear due to the fact that m' is not an integer and depends on the asymptotic values of Φ_0 and χ_0 . On the other hand it is clear that the equation of motion (3.28) possesses constant solution $r = \text{const}$ in case when $n' = 0$ on condition when

$$E^2 = T_{D1}^2 m'^2 = T_{D1}^2 (p^2 + q^2) . \quad (3.36)$$

This is rather puzzling result that shows the difficulty with the background solution (3.1). To see this in more details let us imagine that we have a configuration of the background NS5-brane and probe fundamental string. Under $SL(2, Z)$ transformation these two objects transform differently. Explicitly, since NS5-brane is magnetically charged object with respect to NSNS-two form it transforms in the same way as in (2.4). Then (p, q) –five brane arises from $NS5$ –brane through following $SL(2, Z)$ transformation

$$\begin{pmatrix} \hat{Q}_{NS5} \\ \hat{Q}_{D5} \end{pmatrix} = \begin{pmatrix} p & -c \\ q & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (3.37)$$

so that

$$\Lambda = \begin{pmatrix} a & -q \\ c & p \end{pmatrix} . \quad (3.38)$$

On the other hand we know that the fundamental string transforms under $SL(2, Z)$ transformation as in (2.21). Then we find that for Λ given in (3.38) we obtain (a, c) –string where $(ap + qc = 1)$. Since NS5-brane and fundamental string forms a marginal bound state the previous arguments suggest that such a bound state exists also for (p, q) –five brane and (a, c) –fundamental string. Then the condition given in (2.21) is not in agreement with this claim. We mean that the resolution of this paradox can be found when we construct the background (p, q) –five brane solution with the help of $SL(2, Z)$ transformation rather than procedure used in [8] that was based on the $SL(2, R)$ transformation. This question is now under active investigation and we hope to report our results soon.

3.3 The Case $L \neq 0$

Let us now consider the case of non-zero angular momentum L . Following [14] we rewrite the equation of motion for (3.26) into the form

$$\dot{r}^2 + \frac{A_{(p,q)}^2}{E^2} \left(\frac{L^2}{r^2} + T_{D1}^2 n'^2 \right) + A_{(p,q)} \frac{T_{D1}^2}{E^2} m'^2 - A_{(p,q)} = 0 \quad (3.39)$$

that can be interpreted as the equation of conserved energy for a particle with mass $m = 2$ that moves in the effective potential $V_{eff}(r)$

$$V_{eff} = \frac{A_{(p,q)}^2}{E^2} \left(\frac{L^2}{r^2} + T_{D1}^2 n'^2 \right) + A_{(p,q)} \frac{T_{D1}^2}{E^2} m'^2 - A_{(p,q)} \quad (3.40)$$

with zero energy. Now following [14] we will analyze the behavior of this potential for different values of r . For small r we obtain

$$V_{eff} = \frac{r^2}{Q_{(p,q)}} \left(\frac{L^2}{Q_{(p,q)} E^2} + \frac{T_{D1}^2}{Q_{(p,q)} E^2} m'^2 - 1 \right) . \quad (3.41)$$

On the other hand for large r we have

$$V_{eff} = \frac{T_{D1}^2}{E^2} n'^2 - 1 . \quad (3.42)$$

Now we see that for $E < T_{D1} n'$ the potential V_{eff} approaches positive value for $r \rightarrow \infty$ and since the particle has zero energy we find that it cannot escape to infinity. Further, in order to have trajectories with non-zero r we have to demand that the potential approaches zero from below which implies

$$\frac{L^2}{Q_{(p,q)}} < E^2 - \frac{T_{D1}^2 m'^2}{Q_{(p,q)}} . \quad (3.43)$$

In fact, if this condition were not satisfied that the only solution would be $r = 0$.

Let us now explicitly find the solution of the equation of motion in the throat region when $A_{(p,q)} = \frac{r^2}{Q_{(p,q)}}$. Then the equation (3.26) has the form

$$\dot{r}^2 = \frac{r^2}{Q_{(p,q)}} \left(1 - \frac{T_{D1}^2}{E^2} m'^2 - \frac{L^2}{Q_{(p,q)} E^2} \right) - \frac{r^4}{Q_{(p,q)}^2 E^2} T_{D1}^2 n'^2 \quad (3.44)$$

that has the solution

$$r = \frac{Q_{(p,q)} E}{T_{D1}} \sqrt{1 - m'^2 \frac{T_{D1}^2}{E^2} - \frac{L^2}{Q_{(p,q)} E^2}} \frac{1}{\cosh \sqrt{1 - \frac{T_{D1}^2}{E^2} m'^2 - \frac{L^2}{Q_{(p,q)} E^2}} t} . \quad (3.45)$$

We see that the non-zero angular momentum slows down the decrease of r . Further, the equation of motion for ϕ implies

$$\dot{\phi} = \frac{L}{E Q_{(p,q)}} t . \quad (3.46)$$

In other words, previous solution describes (m', n') -string that moves towards to the world-volume of background five brane which however also circles around them.

As the next example we consider the situation when $n' = 0$. In this case we find the potential in the form

$$V_{eff} = \frac{A_{(p,q)}^2}{E^2} \frac{L^2}{r^2} + A_{(p,q)} \frac{T_{D1}^2}{E^2} m'^2 - A_{(p,q)} \quad (3.47)$$

that in the throat region simplifies as

$$V_{eff} = A_{(p,q)} \left(\frac{L^2}{Q_{(p,q)} E^2} + \frac{T_{D1}^2}{E^2} m'^2 - 1 \right) \quad (3.48)$$

and it vanishes identically when

$$E^2 = \frac{L^2}{Q_{(p,q)}} + T_{D1}^2 m'^2 . \quad (3.49)$$

In other words it is possible to find string that rotates around five brane for any values of r .

The situation is different when $E > T_{D1} n'$ which means that the potential is negative for $r \rightarrow \infty$. Further, if we again have (3.43) we obtain that we approach the point $r = 0$ from below and hence there is no potential barrier. In this case we have a possibility of the particle that starts at $r = 0$ for $t = -\infty$ and it escapes to infinity in time reverse process. On the other hand the situation is different when the bound (3.43) is not satisfied. Let us imagine that we have (m, n) -string initially at large distance from (p, q) -five brane. The probe moves towards to the (p, q) -five brane until it reaches the point when the effective potential vanishes that is at

$$r_{min}^2 = \frac{L^2 - E^2 Q_{(p,q)} - T_{D1}^2 m'^2 Q_{(p,q)}}{E^2 - T_{D1}^2 m'^2 - T_{D1}^2 n'^2} . \quad (3.50)$$

Following [14] we can interpret this process as a scattering of (m, n) -string from the collection of (p, q) -five branes. Since the analysis is completely the same as in [14] we will not repeat it here.

4. (m, n) -String in (p, q) -String Background

In this section we consider dynamics of (m, n) -string in the macroscopic (p, q) -string background [7]

$$\begin{aligned} ds_E^2 &= H_{pq}^{-3/4} [-dt^2 + dy^2] + H_{pq}^{1/4} dx_m dx^m , \quad H_{pq} = 1 + \frac{(2\pi)^6 \alpha'^3 \Delta_{(p,q)}^{1/2}}{r^6 \Omega_7} \equiv 1 + \frac{\alpha}{r^6} , \\ \mathbf{B} &= (\Lambda_{p,q}^{-1})^T \begin{pmatrix} (H_{pq}^{-1} - 1) \\ 0 \end{pmatrix} , \quad \Lambda = \Delta_{(p,q)}^{-1/2} \begin{pmatrix} p - e^{-\Phi_0} q + \chi_0 e^{\Phi_0} (p - q \chi_0) \\ q e^{\Phi_0} (p - q \chi_0) \end{pmatrix} , \\ \mathcal{M} &= \Lambda \begin{pmatrix} H_{pq}^{1/2} & 0 \\ 0 & H_{pq}^{-1/2} \end{pmatrix} \Lambda^T , \quad \Delta_{(p,q)} = e^{-\Phi_0} q^2 + (p - q \chi_0)^2 e^{\Phi_0} . \end{aligned} \quad (4.1)$$

The Hamiltonian density for the time dependent world-sheet modes has the form

$$\mathcal{H}_{fix} = \sqrt{\Pi_m \Pi^m H_{pq}^{-1} + H_{pq}^{-2} T_{D1}^2 (m'^2 + n'^2 H_{pq})} - T_{D1} m' (H_{pq}^{-1} - 1) , \quad (4.2)$$

where \mathbf{m}' is equal to

$$\mathbf{m}' = \begin{pmatrix} m' \\ n' \end{pmatrix} = \Delta_{(p,q)}^{-1/2} \begin{pmatrix} e^{\Phi_0} (p - q\chi_0)(m - \chi_0 n) + e^{-\Phi_0} qn \\ -qm + pn \end{pmatrix} . \quad (4.3)$$

Clearly for $m = p$ and $n = q$ we obtain

$$\mathbf{m}' = \Delta_{(p,q)}^{-1/2} \begin{pmatrix} e^{\Phi_0} (p - q\chi_0)^2 + e^{-\Phi_0} qn \\ 0 \end{pmatrix} = \begin{pmatrix} \Delta_{(p,q)}^{1/2} \\ 0 \end{pmatrix} \quad (4.4)$$

with following physical interpretation. As we know (p, q) -string solution was derived through $SL(2, R)$ transformation from the fundamental string background. Then clearly a fundamental string in a macroscopic string background maps to the same object under $SL(2, Z)$ transformation. On the other hand from (4.4) we see that this is not exactly true since the probe string does not carry integer charge. We again leave the resolution of this paradox for future research.

It is also useful to find components of the stress energy tensor for the (m, n) -string in static gauge. As in the previous section we temporary replace fixed two dimensional metric $\eta_{\alpha\beta}$ with two dimensional metric $\gamma_{\alpha\beta}$ and write the gauge fixed action in the form

$$S_{fixed} = -T_{D1} \int d\tau d\sigma \left(\sqrt{\mathbf{m}^T \mathcal{M}^{-1} \mathbf{m}} H_{pq}^{-3/4} \sqrt{-\det \mathbf{A}} - \sqrt{-\gamma} (\Lambda_{pq}^{-1} \mathbf{m})^T \begin{pmatrix} \frac{1}{H_{pq}} - 1 \\ 0 \end{pmatrix} \right) , \quad (4.5)$$

where

$$\mathbf{A}_{\alpha\beta} = \gamma_{\alpha\beta} + H_{pq} \delta_{mn} \partial_\alpha x^m \partial_\beta x^n . \quad (4.6)$$

Then the components of two dimensional stress energy tensor have the form

$$\begin{aligned} T_{\alpha\beta} &= -\frac{2}{\sqrt{-\gamma}} \frac{\delta S_{fixed}}{\delta \gamma^{\alpha\beta}} = \\ &= -\frac{T_{D1}}{\sqrt{-\gamma}} \gamma_{\alpha\gamma} (\mathbf{A}^{-1})^{\gamma\delta} \gamma_{\delta\beta} \sqrt{-\det \mathbf{A}} \sqrt{\mathbf{m}^T \mathcal{M}^{-1} \mathbf{m}} H_{pq}^{-3/4} + \gamma_{\alpha\beta} (\Lambda_{pq}^{-1} \mathbf{m})^T \begin{pmatrix} \frac{1}{H_{pq}} - 1 \\ 0 \end{pmatrix} . \end{aligned} \quad (4.7)$$

Finally we return back to the flat metric $\gamma_{\alpha\beta} \rightarrow \eta_{\alpha\beta}$ and consider pure time-dependent ansatz. Then we obtain

$$\begin{aligned} T_{\tau\tau} &= \frac{T_{D1} \sqrt{m'^2 + n'^2 H_{pq}}}{H_{pq} \sqrt{1 - H_{pq}(\dot{r}^2 + r^2 \dot{\phi}^2)}} + m' T_{D1} \left(1 - \frac{1}{H_{pq}}\right) , \quad T_{\tau\sigma} = 0 , \\ T_{\sigma\sigma} &= -T_{D1} \sqrt{m'^2 + n'^2 H_{pq}} \frac{1}{H_{pq}} \sqrt{1 - H_{pq}(\dot{r}^2 + r^2 \dot{\phi}^2)} - m' T_{D1} \left(1 - \frac{1}{H_{pq}}\right) , \end{aligned} \quad (4.8)$$

where we also introduced spherical coordinates in the transverse \mathbf{R}^8 space and considered the dynamics of the probe in two dimensional plane with radial variable r and angular ϕ . As a result the Hamiltonian density (4.2) simplifies considerably

$$\mathcal{H}_{fix} = \frac{1}{H_{pq}} \left(\sqrt{H_{pq}(p_r^2 + \frac{1}{r^2}p_\phi^2 + T_{D1}^2 n'^2) + T_{D1}^2 m'^2 + T_{D1} m' (H_{pq} - 1)} \right). \quad (4.9)$$

Note also that the equation of motion for ϕ has the form

$$\dot{\phi} = \{\phi, H_{fix}\} = \frac{L}{r^2(H_{pq}E - T_{D1}m'(H_{pq} - 1))}, \quad (4.10)$$

where we used the fact that $p_\phi = L$ and $\mathcal{H}_{fix} = E$ are conserved. With the help of these results we obtain following components of the stress energy tensor (4.8)

$$\begin{aligned} T_{\tau\tau} &= E, \\ T_{\sigma\sigma} &= \mathcal{P} = -\frac{T_{D1}^2(m'^2 + n'^2 H_{pq})}{H_{pq}(E - T_{D1}m') + T_{D1}m'} - T_{D1}m' \left(1 - \frac{1}{H_{pq}}\right), \end{aligned} \quad (4.11)$$

where \mathcal{P} is the pressure on the world-volume of (m, n) -string.

Let us now proceed to the analysis of dynamics of this probe string. From H_{fix} we derive the equation of motion

$$\dot{r} = \{r, H_{fix}\} = \frac{p_r}{\sqrt{H_{pq}(p_r^2 + \frac{1}{r^2}L^2 + T_{D1}^2 n'^2) + T_{D1}^2 m'^2}}. \quad (4.12)$$

On the other hand from the fact that the energy density is conserved E we can express p_r as

$$p_r^2 = \frac{1}{H_{pq}} \left((H_{pq}(E - T_{D1}m') + m'T_{D1})^2 - T_{D1}^2 m'^2 \right) - \frac{L^2}{r^2} - T_{D1}^2 n'^2 \quad (4.13)$$

so that (4.12) has the form

$$\dot{r}^2 = \frac{1}{H_{pq}} \left(1 - \frac{(T_{D1}^2 n'^2 + \frac{L^2}{r^2})H_{pq} + T_{D1}^2 m'^2}{(H_{pq}(E - T_{D1}m') + T_{D1}m')^2} \right). \quad (4.14)$$

We can again rewrite this equation into the more suggestive form

$$\dot{r}^2 + V_{eff} = 0, \quad (4.15)$$

where

$$V_{eff} = \frac{1}{H_{pq}} \left(\frac{(T_{D1}^2 n'^2 + \frac{L^2}{r^2})H_{pq} + T_{D1}^2 m'^2}{(H_{pq}(E - T_{D1}m') + T_{D1}m')^2} - 1 \right). \quad (4.16)$$

We see that the equation above corresponds to the massive particle with mass $m = 2$ moving in the potential V_{eff} with zero energy so that many interesting information about the particle's trajectory follow from the properties of given potential. As in previous section we start with the case of the zero angular momentum

4.1 The Case $L = 0$

We see that for $r \rightarrow \infty$ we have $H_{pq} \rightarrow 1$ and hence

$$V_{eff} \rightarrow \frac{T_{D1}^2 m'^2 + T_{D1}^2 n'^2}{E^2} - 1 \quad (4.17)$$

while for small r we obtain

$$V_{eff} = \frac{r^6}{\alpha} \left(\frac{n'^2 T_{D1}^2 r^6}{\alpha (E - T_{D1} m')^2} - 1 \right) \quad (4.18)$$

so that V_{eff} approaches the point $r = 0$ from below. As a result we have two qualitative different behaviors of probe (m', n') string in this background. It follows from (4.17) that for $E^2 < T_{D1}^2 (m'^2 + n'^2)$ V_{eff} approaches positive constant for large r . Then the probe string cannot escape to infinity and moves in the bounded region around (p, q) -string background. On the other hand for $E^2 > T_{D1}^2 (m'^2 + n'^2)$ the potential is negative for all values and hence probe string can move to infinity. Let us firstly consider the case when $n' = 0$. This case corresponds to the situation of the motion of fundamental string in the background of collection of the fundamental strings. We can expect that it is possible to form a marginal bound state of $N + m'$ fundamental strings. In fact, for $E - T_{D1} m' = \epsilon \ll 1$ we find that the effective potential has the form

$$V_{eff} = -2 \frac{\epsilon}{T_{D1} m'} \quad (4.19)$$

and we see that it is flat. As a result we find time dependence $r \sim \pm \epsilon t$ which means very slowly movement of the probe string. This is a confirmation of the claim that the probe string can form marginal bound state with N fundamental strings. Note that this approximation is valid on condition when

$$\frac{H_{pq} \epsilon}{T_{D1} m'} \ll 1 \quad (4.20)$$

that implies $r^6 \gg \frac{\alpha \epsilon}{T_{D1} m'}$ that can be obeyed in the whole region in the limit $\epsilon \rightarrow 0$. Finally note also that the pressure is equal to

$$\mathcal{P} = -2T_{D1} m' + H_{pq} \epsilon + \frac{T_{D1}}{H_{pq}} m' \quad (4.21)$$

that has following physical explanation. Consider the initial configuration when the m' -string is sitting at infinity when $H_{pq} = 1$ and consequently $\mathcal{P} = -T_{D1} m' + \epsilon = E - 2T_{D1} m'$. The string moves slowly to the horizon when $H_{pq} \rightarrow \infty$ and hence the pressure approaches the value $\mathcal{P} \rightarrow -m' T_{D1}$ at the horizon in the limit $\epsilon \rightarrow 0$.

To see this more clearly let us consider the case of the near horizon limit when $H_{pq} \epsilon \gg T_{D1} m'$ when ϵ is small but finite. In this case we find that the leading order behavior of the effective potential is

$$V_{eff} = -\frac{r^6}{\alpha} \quad (4.22)$$

and hence we have the differential equation

$$\frac{dr}{dt} = \pm \frac{r^3}{\sqrt{\alpha}} . \quad (4.23)$$

The $+$ sign corresponds to the string moving from the horizon when the approximation we use quickly breaks. The sign $-$ corresponds to the string moving to the horizon and for this possibility we find the solution

$$r = \frac{r_0}{\sqrt{1 + \frac{2r_0^2}{\sqrt{\alpha}}t}} , \quad r_0^6 \ll \alpha . \quad (4.24)$$

We see that the probe string approaches horizon at asymptotic time $t \rightarrow \infty$. Observe that this behavior does not depend on the value of the energy of the string probe.

4.2 The Case $L \neq 0$

Now we would like to analyze the behavior of the potential in case $n' = 0$ and $L \neq 0$ and in the limit $E - T_{D1}m' = \epsilon \ll 1$. In this case we find the effective potential in the form

$$V_{eff} = -2\frac{\epsilon}{T_{D1}m'} + \frac{L^2}{r^2 T_{D1}^2 m'^2} \left(1 - \frac{2\epsilon}{T_{D1}m'}\right) - \frac{2\alpha}{r^8} \frac{L^2 \epsilon}{T_{D1}^3 m'^3} . \quad (4.25)$$

From this form of the effective potential we can deduce an existence of the potential barrier since there are two points where V_{eff} vanishes. We find these points as follows. We presume that the first root corresponds to the root when we neglect term proportional to r^{-8} . Then we solve the quadratic equation with the solution

$$r_+ = \frac{L}{\sqrt{2T_{D1}m'\epsilon}} . \quad (4.26)$$

We see that it has very large value that justifies our assumption. The second root corresponds to the situation when we neglect the constant term in (4.25) and we obtain

$$r_- = \left(\frac{2\alpha\epsilon}{T_{D1}m'} \right)^{1/6} \quad (4.27)$$

that is much smaller than r_+ again with agreement with our assumptions. The physical picture is following. If we have probe m' -string with $E - T_{D1}m' \ll 1$ far away from the background (p, q) -string that it moves towards it however it cannot cross the horizon. Rather it approaches to the distance given by r_+ and than it is deflected. On the other hand m' -string that is initially in the region below r_- will spirally moves towards to the horizon. This situation is similar as in case of $(m, 1)$ -string studied in [15] and we will not repeat it here.

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